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Adjacent Line Graph in Ferrers Graph

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Abstract:

A simple graph $G = (V, E)$ is a Ferrers graph if for all distinct $x, y, z, w \in V$ if $xy \in E$ and $zw \in E$ then either $xw \in E$ or $yz \in E$. In this paper, we study the adjacent line graph of a ferrers graphs. We also check the conditions for adjacent line graphs of Path, Cycle, Complete, Star graphs to be ferrers. Also a sufficient condition has been given for a graph sothat its adjacent line graph is not ferrers.

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1. Introduction

For graph theory notations and terminology not given here we refer it from [1]. The line graph of an undirected graph G is another graph $\mathrm{L}(G)$ that represents the adjacencies between edges of G. Let $G = (V, E)$ be a simple graph with at least one pair of adjacent edges. The adjacent line graph of G, denoted by $AL(G)$, is a graph with the vertex set $V_{AL} = \{v_{ij} / e_i$ and e_j are adjacent in $G\}$ and two vertices v_{ij} and vkl are adjacent in $\mathrm{AL}(G)$, if and only if, either e_i and e_k or e_i and e_l or e_j and e_k or e_j and e_l are adjacent to each other in G. [7].

Ferrers relation was introduced in [2] for the first time has been utilized for different purposes in extensive and various science fields. The relation was used with concept lattices in formal concept analysis. Some graphs associated by the relation were linked together concept lattices again. We already proved some results [4] on Ferrer trees and its distance character. The upper bounds for distance function (u, v) for all $u, v \in V$. Throughout the paper, we consider connected graph.

In this paper our intention is to move a step forward in the investigation of the adjacent line graph of a Ferrers graph and its properties. We also check the conditions for adjacent line graphs of Path, Cycle, Complete, Star graphs to be Ferrers. Also a sufficient condition has been given for a graph so that its adjacent line graph is not Ferrers.

Definition 1.1. [2] A simple graph G is a Ferrers graph if for all distinct $x, y, z, w \in V$ if $xy \in E$ and $zw \in E$ then either $xw \in E$ or $yz \in E$. Since $xy \in E$ if and only if $yx \in E$ holds for all simple graphs, the definition of Ferrers graph must be extended to if $xy \in E$ and $zw \in E$, then either $xw \in E$ or $yz \in E$ or $xz \in E$ or $yw \in E$.

Definition. 1.2. [4] Graphs which do not satisfy the above conditions are classified as non- Ferrers graphs. Also, there are graphs which do not have at least four distinct vertices $x, y, z, w \in G$ such that $xy, zw \in G$. That is, the graph does not exist at least two disjoint edges to verify Ferrers condition. This class of graphs is classified as infringe-Ferrers graphs. The obvious examples are C_3 and P_3. The following Theorems are used in the sequel.

Theorem 1.3. [4] Let G be a tree. Then G is a Ferrers tree if and only if G has two internal vertices.

Theorem 1.4. [4] Let G be a complete graph, then G is a Ferrers graph.
Theorem 1.5. [2] Let $G = (V, E)$ be a simple graph where $|V| \geq 4$. If G is Ferrers, then $(v, w) \leq 2$ for all distinct $v, w \in V$.
Theorem 1.6. [2] Let $G = (V, E)$ is Ferrers if and only if for all distinct $x, y, z, w \in V$ if $xy, zw \in E$ then $\mathrm{d}(x, w) + \mathrm{d}(y, z) \leq 4$.

Theorem 1.7. [7] Let $v_1, v_2, \ldots, v_n$ are vertices of G, then the number of vertices $\mathrm{n_i}$ $AL(G)$ equal to $\sum_{i=1}^{n}\binom{d_i}{2}$,

Where $d_i(\geq 2)$ is the degree of the vertex v_i.

Theorem 1.8. [4] Every n-regular graph in Ferrers if and only if $d(u, v) \leq 2$.

2. Properties of Adjacent line Graph

In this Section we study the Adjacent line graph of a Ferrers graph. Let G be a connected graph of order $n \geq 4$. If the adjacent line graph AL(G) is Ferrers graph, then we call the graph as AL-Ferrers.

Theorem 2.1. Adjacent line graph of a ferrers tree ($n \geq 5, G \neq P_5$) is ferrers.

Proof . Let G is a ferrers tree with $n \geq 5$, G $\neq P_5$. To prove AL(G) is ferrers

Since G is a ferrers tree, it has 2 internal vertices, and n end vertices. Let u and v be an 2 internal vertices and $e = uv$ is the edge. Then the edge "e" is incident with every edges of G. Therefore, e is an universal vertex of AL(G). Which implies d($AL(G) \leq 2$). Then for four distinct vertices $w, x, y, z \in$ AL(G). Then d(w, x) + d(y, z) $\leq 2 + 2 = 4$. By Theorem 1.6, AL(G) is ferrers.

Remark 2.2. For $n = 4$, AL(P_4) = P_2, It is an Infringe Ferrers graph.
For $n = 5$, AL(P_5) = P_3, It is an Infringe Ferrers graph.

Theorem 2.3. For a ferrers tree G, AL(G) is ferrers, diam $AL(G)$ =2 and rad $AL(G)$=1.

Proof. Consider a ferrers tree G and AL(G) is ferrers. To prove diam AL(G) = 2. Suppose diam AL(G) ≥ 2. The G isa non- ferrers tree (or) a path containing atleast vertices. In all the cases G is not a ferrers tree, which is a contradiction. Hence $diam$ AL(G) = 2. Now to prove that, rad AL(G) is 1. It is enough to prove that rad AL(G) $\neq 2$ Since G is a ferrers tree, the internal vertex edge $e = uv$ is incident to all other vertices. Then clearly rad AL(G) = 1.

Theorem 2.4. Adjacent line graph of a non-Ferrers tree which is not a path is Ferrers if G has 3 internal vertices.

Proof. Consider a adjacent line graph of a non-Ferrers tree, which is Ferrers. We have to prove that, G has 3 internal vertices. Suppose G has more than 3 internal vertices. Then AL(G) is a non-Ferrers graph also $d(u, v) > 3 \;\forall\; u, v \in AL$(G), which a contradiction to our assumption. Hence G has 3 internal vertices.

Theorem 2.5. The Adjacent Line Graph of G is not Ferrers if and only if the graph G contains two disjoint subgraphs of either P_4 or $K_{1,3}$ or C_3.

Proof. Let G be a graph and AL(G) be the adjacent line graph of G. Let P and Q be the two subgraphs of G which are disjoint. Suppose P and Q are either P_4 or $K_{1,3}$ or C_3. In all the three subgraphs, have exactly three edges. Let e_1, e_2, e_3 and e_x, e_y, e_z be the distinct edges of P and Q respectively. Let v_{12}, v_{13}, v_{23} and v_{xy}, v_{yz}, v_{xz} be the six distinct vertices in AL(G). Clearly $v_{12}v_{13}$ and $v_{xy}v_{xz}$ are two distinct edges in AL(G). Since P and Q are disjoint, by the definition of adjacent line graph by there does not exist any edge $v_{12}v_{xy}$ or $v_{12}v_{xz}$ or $v_{xy}v_{13}$ or $v_{xz}v_{13}$. Hence by the definition of adjacent line graph, AL(G) is a non-Ferrers graph and hence G is not AL-Ferrers.

Conversely, Assume that the adjacent line graph of G is a non-Ferrers graph. Then by the definition of a Ferrers graph, there exists at least two distinct edges $v_{12}v_{23}$ and $v_{xy}v_{yz}$ such that there does not exists edges $v_{12}v_{xy}$ or $v_{12}v_{yz}$ or $v_{xy}v_{13}$ or $v_{yz}v_{13}$ in AL(G) and hence the edges $v_{12}v_{23}$ and $v_{xy}v_{yz}$ are non-adjacent to each other in AL(G). Now $v_{12}v_{23}$ is an edge in AL(G) implies that e_1, e_2, e_3 are all edges in G which are adjacent to each other. Now the simple connected graphs with three edges are either P_4 or $K_{1,3}$ or C_3 in G. Similarly $v_{xy}v_{yz}$ is an edge in AL(G) implies that e_x, e_y, e_z are all edges in G which are adjacent to each other and hence the connected graphs with three edges are either P_4 or $K_{1,3}$ or C_3 in G. Let the two subgraphs obtained from the edges $v_{12}v_{23}$ and $v_{xy}v_{yz}$ be P and Q in G. To prove P and Q are disjoint.

Claim P and Q are disjoint.

Suppose P and Q are connected, then there exist an edge between P and Q in G. Without loss of generality let e_m be the edge between P and Q. Then clearly there exists a vertex v_{ml} such that v_{ml} will be either adjacent to either v_{12} or v_{23} or v_{xy} or v_{yz}. Then by the definition of adjacent line graph, AL(G) is a Ferrers graph. This is a contradiction to AL(G) is a non-Ferrers graph. Thus P and Q are disjoint. Hence the proof.

3. Adjacent Line Graph of Path, Cycle, Complete, Star Graphs to be Ferrers

Theorem 3.1. For a path, $AL(P_n)$ is a Ferrers graph if and only if n is either 6 or 7.

Proof. Let $AL(P_n)$ denotes the adjacent line graph of a path with n vertices.

Case 1. $n \le 5$, For $n = 2$, $AL(P_2)$ is not possible as P_2 has only one edge.

If n = 3, 4 and 5 then their respective adjacent line graphs are $AL(P_3) = K_1, AL(P_4) = K_2$ and $AL(P_5) = P_3$ respectively which are all infringe-Ferrers graphs, which is a contradiction to G is a Ferrers graph.

Case 2. n = 8

Let $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ and v_8 be the vertices of P_8 and $e_1, e_2, e_3, e_4, e_5, e_6$ and e_7 be the edges of P_8. By the definition of adjacent line graph, $v_{12}, v_{23}, v_{34}, v_{45}, v_{56}$ and v_{67} be the vertices of $AL(P_8)$ and its edge set is given by $v_{12}v_{23}, v_{23}v_{34}, v_{34}v_{45}, v_{45}v_{56}, v_{12}v_{34}, v_{56}v_{34}, v_{45}v_{23}, v_{67}v_{45}, v_{67}v_{56}$ and is shown in Figure 1. Clearly the edges $v_{67}v_{56}$ and $v_{12}v_{23}$ are non-adjacent to each other. Hence $AL(P_8)$ is a non-Ferrers graph.

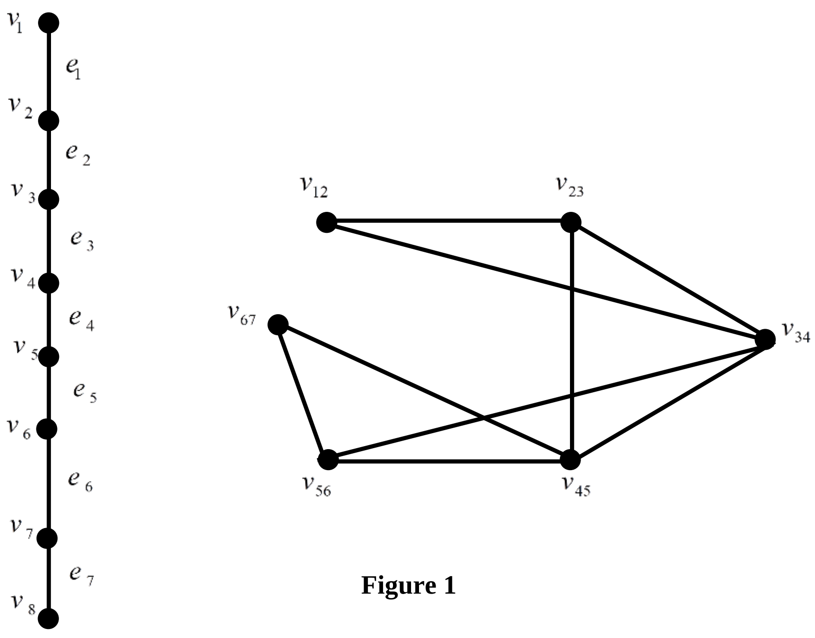

Figure 1

Case 3. $n \geq 9$

Clearly G has at least 9 vertices. Therefore we get two disjoint P_4. By Theorem 2.5, $AL(P_n)$ is not Ferrer for $n \geq 9$.

Hence from the above three cases it is cases it is clear that (P_n) is Ferrers if n is either 6 or 7.

Conversly, Consider P_6 and P_7. Now we prove that $AL(P_n)$ is Ferrer for n = 6 and 7.

Let us consider the following.

Case 1. n = 6

Let v_1, v_2, v_3, v_4, v_5 and v_6 be the vertices of P_6 and e_1, e_2, e_3, e_4, e_5 be the edges of P_6. By the definition of the adjacent line graph, $v_{12}, v_{23}, v_{34}, v_{45}, v_{56}$ be the vertices of $AL(P_6)$ and its edge set is given by $v_{12}v_{23}, v_{23}v_{34}, v_{34}v_{45}, v_{12}v_{34}, v_{45}v_{23}$ and is shown in Figure 2 and it can be clearly observe that $d(u,v) \leq 2$; for all $u, v \in V(AL(P_6))$. Hence by Theorem 1.8, $AL(P_6)$ is Ferrers.

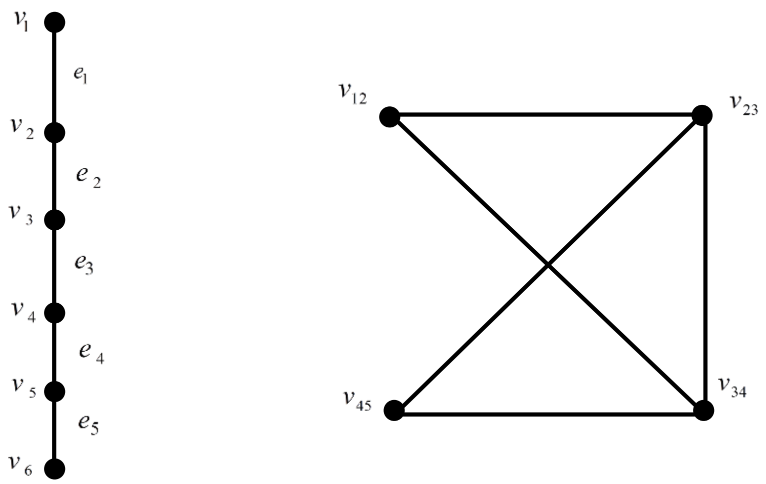

Figure 2

Case 2. n = 7

Let $v_1, v_2, v_3, v_4, v_5, v_6$ and v_7 be the vertices of P_7 and e_1, e_2, e_3, e_4, e_5 and e_6 be the edges of P_7. By the definition of adjacent line graph, $v_{12}, v_{23}, v_{34}, v_{45}$ and v_{56} are the vertices of $AL(P_7)$ and its edge set is given by $v_{12}v_{23}, v_{23}v_{34}, v_{34}v_{45}, v_{45}v_{56}, v_{12}v_{34}, v_{56}v_{23}$ and is shown in Figure 3 and it can be clearly observe that $d(u,v) \leq 2$; for all $u, v \in V(AL(P_7))$. Hence by Theorem 1.8, $AL(P_7)$ is Ferrers.

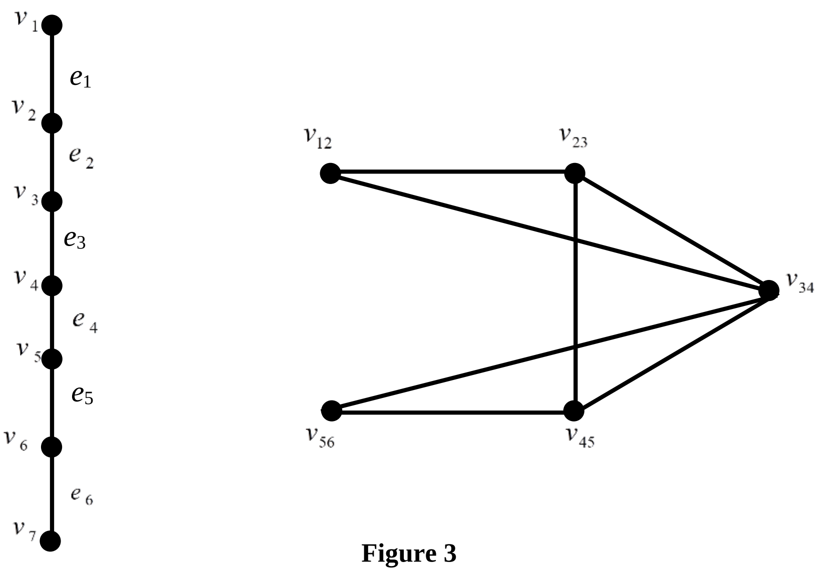

Figure 3

Theorem 3.2. Adjacent line graph of the cycle is Ferrers if and only if n = 4, 5, 6 and 7.

Proof. Let $AL(C_n)$ denotes the adjacent line graph of a cycle with n vertices. We consider the following cases

Case 1. n = 3

Clearly $AL(C_3)=C_3$ which is an infringe Ferrer graph.

Case 2. n = 4

Let $C_4 = v_1e_1v_2e_2v_3e_3v_4e_4v_1$. Then the vertices of $AL(C_n)$ are denoted by $v_{12}, v_{23}, v_{14}, v_{34}$. Consider the vertex $v_{i(i+1)}$ in $AL(C_n)$. Then $v_{i(i+1)}$ it is adjacent to all the remaining three vertices $v_{(i+1)(i+2)}, v_{(i+2)(i+3)}, v_{i(i+3)}$ and hence the degree of $v_{i(i+1)}$ is 3. Similarly v_{23} is adjacent to v_{12}, v_{14}, v_{13} and v_{14} is adjacent to v_{12}, v_{23}, v_{34}. And v_{34} is adjacent to v_{12}, v_{23}, v_{14}. Thus all the vertices of $AL(C_n)$ where $n = 4$ of degree 3, which is complete. By Theorem 1.4, hence $AL(C_4)$ is Ferrers.

Case 3. $n \geq 5$

Let $C_n = v_1e_1v_2e_2, ..., e_{i-1}v_ie_iv_{i+1}...e_{n-1}v_ne_nv_1$. Then the vertices of $AL(C_n)$ are denoted by $v_{12}, v_{23}, v_{34}, ..., v_{ni}$. Consider the vertex $v_{i(i+1)}$ in $AL(C_n)$, it is adjacent to the 4 vertices $v_{(i-2)(i-1)}, v_{(i-1)i}, v_{(i+1)(i+2)}$ and $v_{(i+2)}v_{(i+3)}$. So the degree of the vertex $v_{(i+1)}$ is 4. Similarly the vertex v_{12} is adjacent to $v_{23}, v_{34}, v_{nl,} v_{(n-1)n}$ and v is adjacent to $v_{12}, v_{23}, v_{(n-1)n}, v_{(n-1)(n-2)}$ Thus all the vertices of $AL(C_n)$ are of degree 4. Hence $AL(C_n)$ is 4-regular.

Case 3.a. $5 \leq n \leq 7$

Clearly for C_5, $AL(C_5)$ is a 4-regular graphs with 5 vertices and hence we have $d(u,v)=1$. By Theorem 1.4, $AL(C_5)$ is Ferrer. Similarly for C_6, $AL(C_6)$ is a 4-regular graph with 6 vertices and hence we have $d(u,v)=2$. By Theorem 2.3, $AL(C_6)$ is 4-regular graph with 6 vertices and hence we have $d(u,v)=2$. By Theorem 2.3, $AL(C_6)$ is Ferrers. Also for C_7, $AL(C_7)$ is 4-regular graph with 7 vertices and hence we have $d(u,v)=2$. By Theorem 2.3, $AL(C_7)$ is Ferrers.

Thus from the above cases we get, $AL(C_n)$ is Ferrers only if $4 \le n \le 7$.

Thus C_n, is AL-Ferrer for n = 4, 5, 6 and 7.

Case 3.b. $n \ge 8$

Let $C_n = v_1 e_1 v_2 e_2, \ldots, e_{i-1} v_1 e_i v_{i+1} \ldots e_{n-1} v_n e_n v_1$. Then the vertices of $AL(C_n)$ are denoted by $v_{12}, v_{23}, v_{34}, \ldots, v_{ni}$ Consider the vertex $v_{i(i+1)}$ in $AL(C_n)$, it is adjacent to the 4 vertices $v_{(i-2)(i-1)}, v_{(i-1)i}, v_{(i+1)(i+2)}$ and $v_{(i+2)(i+3)}$. So the degree of the vertex $V_{i(i+1)}$ is 4. Similarly the vertex v_{12} is adjacent to $v_{23}, v_{34}, v_{nl}, v_{(n-1)n}$ and v is adjacent to $v_{12}, v_{23}, v_{(n-1)n}, v_{(n-1)(n-2)}$. Thus all the vertices of $AL(C_n)$ are of degree 4. Hence $AL(C_n)$ is 4-regular. For $n \ge 8$, $AL(C_n)$ is 4-regular graph with n vertices and $d(u,v) = n-4$. By Theorem 1.8, every n-regular graph is Ferrers if and only if $d(u,v) \le 2$. But for $n \ge 8$, $d(u,v) = n-4 > 3$. This is a contradiction. Hence for $n \ge 8$, $AL(C_n)$ is not Ferrers.

Hence AL(C_n) is non-Ferrers only if $n \ge 8$.

Hence the theorem.

Lemma 3.3. If $G = K_n$, then $AL(G) = K_p$, where $p = \dfrac{n(n-1)(n-2)}{2}$

Proof. Let G be a complete graph of order n. Clearly G has nC_2 edges in G. Let $AL(G)$ denote the adjacent line graph of K_n. By Theorem 1.7, the number of vertices of $AL(G)$ is equal to $\sum_{i=1}^{n} \binom{d_i}{2}$ where $d_i (\ge 2)$ is the degree of the vertex v_i. Since every vertex is of degree $n-1$ in G, we have the number of vertices in $AL(G) = \sum_{i=1}^{n} \binom{n-1}{2} = \dfrac{\{n(n-1)(n-2)\}}{2}$. Hence $AL(G)$ is of order $\dfrac{\{n(n-1)(n-2)\}}{2}$.

Let $\dfrac{\{n(n-1)(n-2)\}}{2} = p$. G is a complete graph implies that any two vertices are adjacent to each other. This implies that for any two distinct to each other v_{ij} and v_{kl}, in $AL(G)$, there exist anedge between them as either e_i and e_k or e_j and e_l or e_i and e_l or e_j and e_k or e_i and e_k are adjacent to each other. Hence by the definition of adjacent line graph, there exist an edge between any two vertices. Since $AL(G)$ has P vertices, $AL(G) = K_p$, where $P = \dfrac{\{n(n-1)(n-2)\}}{2}$

Theorem 3.4. $AL(K_n)$ is Ferrer.

Proof. By Lemma 3.3, we have $AL(K_n) = K_P$, where $p = \dfrac{[n(n-1)(n-2)]}{2}$. Also by Theorem 1.4, hence $AL(K_n)$ is Ferrers.

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Theorem 3.5. For a star graph $K_{1,n}.AL(K_{1,n})$ is Ferrer if and only if $n \geq 4$.

Proof. A star graph is the complete bipartite graph $K_{1,n}$. Let $AL(K_{1,n})$ denote the adjacent line graph of the star graph $K_{1,n}$. Let the adjacent line graph of star be Ferrers. That is $AL(K_{1,n})$ is Ferrers graph. Now we prove that n ≥ 4. Suppose $n=1$, then $K_{1,1}=P_2$ which implies that $AL(P_2)$ is not possible and hence it does not exist. If n = 2, then $K_{1,2}=P_3$ which implies that $AL(P_3)=P_2$ which is an infringe Ferrer graph. Similarly if n = 3, then $G=K_{1,3}$ which implies that $AL(K_{1,3})=C_3$ which is also an infringe Ferrer graph. Thus $K_{1,n}$ is not Ferrer for $n \leq 3$ which implies that $n \geq 4$.

Conversely, consider the star graph $K_{1,n}$, $n \geq 4$. Now we prove that $AL(K_{1,n})$ is Ferrer. Let u be the central vertex and $v_1, v_2, v_3, \ldots, v_n$ be the pendent vertices of $K_{1,n}$. Clearly the degree of the central vertex is n and all other vertices are of degree one as they are pendent vertices. Now the number of vertices in $AL(K_{1,n})$ is given by nC_2. Since each edge is adjacent with all other edges in $K_{1,n}$, every vertex in $AL(K_{1,n})$ is adjacent with every other vertices of $AL(K_{1,n})$. Hence degree of each vertex in $AL(K_{1,n})$ is $nC_2 - 1$. Hence the adjacent line graph of $K_{1,n}$ is the complete graph K_{nC_2}. By Theorem 1.4, we know that every complete graph is Ferrer and hence $AL(K_{1,n}) = K_{nC_2}$, is Ferrers. Thus we have $K_{1,n}$ is AL-Ferrer if and only if $n \geq 4$.

Lemma 3.6. If $G=K_{m,n}$, where m, n > 1 then $AL(G)$ is complete.

Proof. Let $G=K_{m,n}$, where m, n > 1 be a complete bi-partite graph. Let $AL(G)$ denote the adjacent line graph of $K_{m,n}$. Since G is a bi-partite graph. Let $v_1, v_2, \ldots, v_n$ be the vertices in V_1 and $u_1, u_2, \ldots, u_n$ be the vertices in V_2. Since we have, every vertices in V_1 are adjacent to V_2 and every vertices in V_2 are adjacent to V_1. By Theorem 1.7, hence the number of vertices in $AL(G)=\sum_{i=1}^{m}\binom{d_i}{2}+\sum_{j=1}^{n}\binom{d_j}{2}$. Since each edge is incident with all the vertices in both V_1 and V_2. By the definition of adjacent line graph, each vertex in $AL(G)$ is adjacent to the remaining vertices. In this way we get AL(G) is complete.

Theorem 3.7. $AL(K_{m,n})$ is a Ferrers.

Proof. By Lemma 3.6, $AL(K_{m,n})$ is complete. Also by Theorem 1.4, hence $AL(K_{m,n})$ is a Ferrers graph.

4. Conclusion

In this article, we have studied the adjacent line graph of a Ferrers graphs. We also verified that the conditions for adjacent line graphs of path, Cycle, Complete, Star graph to be Ferrers. Also a sufficient condition has been given for a graph so that its adjacent line graph is not Ferrers.

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